

# Revisit the $\Lambda$ CDM Universe in $f(R)$ gravity

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## Abstract

We perform a number of explicit reconstructions for the  $f(R)$  gravity model to give rise to the particular  $\Lambda$ CDM evolution of the universe. We find well-defined real valued analytical forms for the  $f(R)$  model to describe the universe both in the early epoch from radiation to matter dominated era and the late time acceleration period. We further examine the viability of the derived  $f(R)$  model and find that it is viable to describe the universe evolution in the past and there does not exist the future singularity in the Lagrange.

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There has been accumulated conclusive evidence from supernovae[1] and other observations[2, 3] in the last decade, indicating that our Universe is undergoing a phase of accelerated expansion. Understanding the nature of the cosmic acceleration is one of the biggest questions in modern physics. The leading explanation of the accelerated expansion is the cosmological constant in the context of General Relativity. However, the measured value of cosmological constant is far below the prediction of any sensible quantum field theories and furthermore the cosmological constant leads inevitably to the coincidence problem, namely why the energy densities of matter and the vacuum are in the same order today(see [4] for review).

Alternatively, another possible explanation of the acceleration is the modified gravity. The General relativity might not be ultimately correct in the cosmological scale. The Universe might be described by some kind of modified gravity. One simplest attempt is called  $f(R)$  gravity, in which the scalar curvature in the Lagrange density of Einstein's gravity is replaced by an arbitrary function of  $R$ . However the complexity of the field equations makes it difficult to obtain a viable  $f(R)$  model to satisfy both cosmological and local gravity constraints[5]. Recently, there appeared a useful approach to reconstruct the  $f(R)$  model from the observed expansion history of the universe and try to invert the field equations to deduce what class of  $f(R)$  theories give rise to the particular cosmological evolution [6][7][8][9][10][11]. Some analytical forms for  $f(R)$  gravity to admit the  $\Lambda$ CDM expansion history in the background spacetime were constructed [9] [10]. However, it was argued that only simple real expression of  $f(R)$  model in the Lagrange could admit an exact  $\Lambda$ CDM expansion history[10].

In this paper, we will further study this problem. We will perform a number of explicit reconstructions which lead to a number of interesting results. We will show that we can derive a well-defined real analytical  $f(R)$  form in terms of the hypergeometric functions to admit an exact  $\Lambda$ CDM expansion history. We will disclose that the  $f(R)$  gravity not only can admit an exact  $\Lambda$ CDM expansion in the recent epoch of the Universe but also can admit an exact  $\Lambda$ CDM expansion in the early time of the Universe.

We consider the background dynamics of the Universe in  $f(R)$  gravity described by[12]

$$H^2 = \frac{FR - f}{6F} - H \frac{\dot{F}}{F} + \frac{\kappa^2}{3F} \rho \quad . \quad (1)$$

where  $\rho = \rho_m + \rho_r$ ,  $F = \frac{\partial f(R)}{\partial R}$  and  $\kappa^2 = 8\pi G$ . If we convert the derivative from the cosmic

time  $t$  to  $x = \ln a$  and further take the derivative of the above equation, we obtain

$$\frac{d^2}{dx^2}F + \left(\frac{1}{2}\frac{d\ln E}{dx} - 1\right)\frac{dF}{dx} + \frac{d\ln E}{dx}F = \frac{\kappa^2}{3E}\frac{d\rho}{dx} \quad , \quad (2)$$

where

$$\begin{aligned} E &\equiv \frac{H^2}{H_0^2} \quad , \\ R &\equiv 3\left(\frac{dE}{dx} + 4E\right) \quad , \\ \frac{d\rho}{dx} &= -3(\rho + p) \quad . \end{aligned} \quad (3)$$

For convenience, we take the energy density  $\rho$  and the scalar curvature  $R$  in Eq.(2) in the unit of  $H_0^2$  and we also set  $\kappa^2 = 1$  in our analysis. In order to get a viable  $f(R)$  model with a reasonable expansion history of the Universe allowed by observations, we can parameterize  $E(x)$  in Eq.(2) as the standard model in Einstein's gravity with an effective dark energy equation of state(EoS)  $w$  [6][7]

$$E(x) = \Omega_r^0 e^{-4x} + \Omega_m^0 e^{-3x} + \Omega_d^0 e^{-3\int_0^x (1+w)dx} \quad , \quad (4)$$

where

$$\begin{aligned} \Omega_m^0 &\equiv \frac{\kappa^2 \rho_m^0}{3H_0^2} \quad , \\ \Omega_d^0 &\equiv \frac{\kappa^2 \rho_d^0}{3H_0^2} \quad , \\ \Omega_r^0 &\equiv \frac{\kappa^2 \rho_r^0}{3H_0^2} \quad . \end{aligned} \quad (5)$$

After specifying the expansion history of the Universe  $E(x)$ , Eq.(2) becomes a second order differential equation of  $F(x)$ . If we can find the solution of Eq.(2), we then obtain the explicit form of  $f(R)$  correspondingly. We find that it is more convenient to use the quantity  $G = F - 1$  instead of  $F$ , so that Eq.(2) can be changed into

$$\frac{d^2 G}{dx^2} + \left(\frac{1}{2}\frac{d\ln E}{dx} - 1\right)\frac{dG}{dx} + \frac{d\ln E}{dx}G = \frac{3(1+w)\Omega_d^0}{E} e^{-3\int_0^x (1+w)dx} \quad . \quad (6)$$

To mimic the exact  $\Lambda$ CDM expansion history of the universe with  $w = -1$ , Eq.( 6) is a homogenous equation. We will focus on this case hereafter and show that the solution of the above differential equation will directly lead to the real valued  $f(R)$  form which gives rise to the cosmological evolution as that of the  $\Lambda$ CDM model.

Eq.(6) does not have the overall analytical solution with the full expression of Eq.(4) to describe the universe from the radiation dominated epoch to the late time acceleration. However, Eq.(6) does have analytical solutions in different epochs in the evolution of the Universe. Let's first concentrate on the early evolution of the Universe from the radiation dominated epoch to the matter dominated epoch. In this case,  $E$  can be taken as

$$E_1 \sim \Omega_m^0 e^{-3x} + \Omega_r^0 e^{-4x} \quad ,$$

and the general solution of Eq.(6) has the form

$$G_1(x) = C_1 G_{22}^{22} \left( \begin{matrix} n_- & n_+ \\ -1 & 4 \end{matrix} \middle| -\frac{\Omega_m^0}{\Omega_r^0} e^x \right) + D_1 e^{4x} {}_2F_1[m_-, m_+; 6; -\frac{\Omega_m^0}{\Omega_r^0} e^x] \quad (7)$$

where  $G_{22}^{22}$  is the Meijer G function,  ${}_2F_1$  is the Gaussian hypergeometric function and  $C_1, D_1$  are arbitrary constants which can be determined by boundary conditions.

The indexes in the solutions are

$$\begin{aligned} m_+ &= \frac{11 + \sqrt{73}}{4} \quad , \\ m_- &= \frac{11 - \sqrt{73}}{4} \quad , \\ n_+ &= \frac{9 + \sqrt{73}}{4} \quad , \\ n_- &= \frac{9 - \sqrt{73}}{4} \quad . \end{aligned}$$

The viable  $f(R)$  models should be in a “chameleon” type [13][14] which provides a mechanism to pass the local test. However, the first term of Eq.(7) is divergent when  $x$  goes to the minus infinity. Thus the requirement  $\lim_{x \rightarrow -\infty} G(x) = 0$  puts the condition  $C_1 = 0$ , so that  $G_1(x)$  turns out to be

$$G_1(x) = D_1 e^{4x} {}_2F_1[m_-, m_+; 6; -\frac{\Omega_m^0}{\Omega_r^0} e^x]. \quad (8)$$

We can obtain the explicit form for the  $f(R)$  model to describe the early universe by doing the integration

$$f(R) = R + \int G(x) \frac{dR}{dx} dx \quad , \quad (9)$$

where the scalar curvature  $R$  can be written as

$$R \rightarrow 3\Omega_m^0 e^{-3x} \quad , \quad (10)$$

which only contains the component of matter because the radiation does not have any contribution to the scalar curvature  $R$  since its energy momentum tensor is traceless. Eq.(10) is valid in the expansion history of the Universe from the radiation dominated epoch to the deep matter dominated epoch. The term  $\frac{dR}{dx}$  in Eq.(9) thus can be expressed as

$$\frac{dR}{dx} = -9\Omega_m^0 e^{-3x} \quad ,$$

where  $x$ , in turn, can be presented in terms of  $R$

$$x(R) = -\frac{1}{3} \ln \left( \frac{R}{3\Omega_m^0} \right) \quad . \quad (11)$$

When  $x$  goes to the minus infinity  $x \rightarrow -\infty$ ,  $R$  goes to infinity  $R \rightarrow +\infty$ . Combing the above equations, we obtain the explicit expression for  $f(R)$  as

$$f_1(R) = R - \frac{45\Omega_r^0 D_1}{(m_+ - 1)(m_- - 1)} \times \left\{ 1 - {}_2F_1 \left[ m_- - 1, m_+ - 1; 5; -\frac{\Omega_m^0}{\Omega_r^0} \left( \frac{3\Omega_m^0}{R} \right)^{1/3} \right] \right\} \quad , \quad (12)$$

where  $f_1(R)$  and  $R$  are in the unite of  $H_0^2$ . Eq.(12) is mathematically well-defined for all the positive value of the scalar curvature  $R > 0$  because the hypergeometric function  ${}_2F_1[a, b; c; z]$  has the integral representation on the real axis when  $b > 0$  and  $c > 0$

$${}_2F_1[a, b; c; z] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} dt \quad , \quad (13)$$

where  $\Gamma$  is the Euler Gamma function. The above expression is well-defined in the range  $-\infty < z < 1$  and the resulting value of  ${}_2F_1[a, b; c; z]$  is also a real value. In order to present the  $f(R)$  in SI units, we can insert Eq.(5) and then we obtain

$$f_1(R) = R - \frac{15\kappa^2 \rho_r^0 D_1}{(m_+ - 1)(m_- - 1)} \left\{ 1 - {}_2F_1 \left[ m_- - 1, m_+ - 1; 5; -\frac{\rho_m^0}{\rho_r^0} \left( \frac{R_0 - 4\Lambda}{R} \right)^{1/3} \right] \right\} \quad . \quad (14)$$

where  $\rho_r^0$  and  $\rho_m^0$  are the energy density of radiation and matter at today respectively.  $R_0$  is the scalar curvature  $R_0 = \kappa^2 \rho_m^0 + \kappa^2 \rho_r^0 + 4\Lambda$ . Eq.(14) shows that we have the well-defined real analytical function for  $f(R)$  gravity that can exactly reproduce the same background expansion as that of the  $\Lambda$ CDM model from the radiation dominated epoch to the matter dominated epoch. At very early time when the curvature is very high,  $R \gg 4\Lambda$ , we have

$${}_2F_1 \left[ m_- - 1, m_+ - 1; 5; -\frac{\rho_m^0}{\rho_r^0} \left( \frac{R_0 - 4\Lambda}{R} \right)^{1/3} \right] \approx 1 - \frac{1}{5} (m_- - 1)(m_+ - 1) \frac{\rho_m^0}{\rho_r^0} \left( \frac{R_0 - 4\Lambda}{R} \right)^{1/3} \quad . \quad (15)$$

Thus, Eq.(14) reduces to

$$f_1(R) \sim R - D_1 3\kappa^2 \rho_m^0 \left( \frac{\kappa^2 \rho_m^0}{R} \right)^{1/3} \quad (16)$$

which shows that in the high curvature region  $R \gg \kappa^2 \rho_m^0$ ,  $f_1(R)$  goes back to the standard Einstein's model  $f_1(R) \sim R$ .

Next we turn to investigate the most interesting case and try to answer whether the  $f(R)$  model can mimic the evolution of the Universe from the matter dominated epoch to the late time acceleration. In this case,  $E$  can be taken as

$$E_2 = \Omega_m^0 e^{-3x} + \Omega_d^0 \quad ,$$

where  $\Omega_d^0$  is a constant. The scalar curvature can be presented as

$$R = 3\Omega_m^0 e^{-3x} + 12\Omega_d^0 \quad . \quad (17)$$

The general solution of Eq.(6) gives

$$G_2(x) = C_2 (e^{3x})^{p_-} {}_2F_1[q_-, p_-; r_-; -e^{3x} \frac{\Omega_d^0}{\Omega_m^0}] + D_2 (e^{3x})^{p_+} {}_2F_1[q_+, p_+; r_+; -e^{3x} \frac{\Omega_d^0}{\Omega_m^0}] \quad , \quad (18)$$

where the indexes are

$$\begin{aligned} q_+ &= \frac{1 + \sqrt{73}}{12} \quad , \\ q_- &= \frac{1 - \sqrt{73}}{12} \quad , \\ r_+ &= 1 + \frac{\sqrt{73}}{6} \quad , \\ r_- &= 1 - \frac{\sqrt{73}}{6} \quad , \\ p_+ &= \frac{5 + \sqrt{73}}{12} \quad , \\ p_- &= \frac{5 - \sqrt{73}}{12} \quad . \end{aligned}$$

After doing integration, we can get the explicit expression for  $f(R)$  from Eq.(9)

$$f_2(R) = R - 6\Omega_d^0 + \epsilon_- + \epsilon_+ \quad , \quad (19)$$

where

$$\epsilon_- = \frac{C_2}{p_- - 1} \left( \frac{3\Omega_m^0}{R - 12\Omega_d^0} \right)^{p_- - 1} \times {}_2F_1 \left[ q_-, p_- - 1; r_-; -\frac{3\Omega_d^0}{R - 12\Omega_d^0} \right] \quad .$$

$$\epsilon_+ = \frac{D_2}{p_+ - 1} \left( \frac{3\Omega_m^0}{R - 12\Omega_d^0} \right)^{p_+ - 1} \times {}_2F_1 \left[ q_+, p_+ - 1; r_+; -\frac{3\Omega_d^0}{R - 12\Omega_d^0} \right] .$$

Inserting Eq.(5), we obtain

$$\begin{aligned} f_2(R) = R - 2\Lambda - \varpi_1 \left( \frac{\Lambda}{R - 4\Lambda} \right)^{p_+ - 1} {}_2F_1 \left[ q_+, p_+ - 1; r_+; -\frac{\Lambda}{R - 4\Lambda} \right] \\ - \varpi_2 \left( \frac{\Lambda}{R - 4\Lambda} \right)^{p_- - 1} {}_2F_1 \left[ q_-, p_- - 1; r_-; -\frac{\Lambda}{R - 4\Lambda} \right] \end{aligned} \quad (20)$$

where  $\varpi_1 = D_2(R_0 - 4\Lambda)^{p_+}/(p_+ - 1)/\Lambda^{p_+ - 1}$  and  $\varpi_2 = C_2(R_0 - 4\Lambda)^{p_-}/(p_- - 1)/\Lambda^{p_- - 1}$ . The constant parameter  $\Lambda$  is defined as  $\Lambda \equiv \kappa^2 \rho_d$  and  $\rho_d$  is the effective energy density of dark energy. When  $\varpi_1 = \varpi_2 = 0$ ,  $\Lambda$  is just the cosmological constant. Noting the fact that  $p_- - 1 < 0$ , when  $R \rightarrow +\infty$ , the second term of Eq.(20) becomes divergent. This is not allowed for the “chameleon” type solution, so that we need to set  $\varpi_2 = 0$ . Therefore the solution has the form

$$f_2(R) = R - 2\Lambda - \varpi_1 \left( \frac{\Lambda}{R - 4\Lambda} \right)^{p_+ - 1} {}_2F_1 \left[ q_+, p_+ - 1; r_+; -\frac{\Lambda}{R - 4\Lambda} \right]. \quad (21)$$

From the integral representation of the hypergeometric function Eq.(13), it is clear that Eq.(21) is mathematically well-defined in the range  $R > 4\Lambda$ . Eq.(21) is a real function in the physical range from the matter dominated epoch to the future expansion of the Universe.

Eq.(6) is our starting point to find the analytic expression for the  $f(R)$  model to mimic the  $\Lambda$ CDM cosmology. From Eq.(17), we can see clearly that the scalar curvature  $R$  obtained from Eq.(6) can be constrained automatically in the physical range  $R > 4\Lambda$ . Our starting point is different from that in [9] [10], where they got their solution by solving the differential equation [9] [10] [11]

$$-3(R - 3\Lambda)(R - 4\Lambda)\frac{d^2}{dR^2}f(R) + \left(\frac{R}{2} - 3\Lambda\right)\frac{d}{dR}f(R) + \frac{1}{2}f(R) + 4\Lambda - R = 0 \quad (22)$$

This equation was derived from Eq.(1). In Eq.(22),  $R$  can be chosen as any value on the real axes, so that not all solutions of Eq.(22) are physical. We need carefully to analyze the solutions of Eq.(22).

A particular solution for Eq.(22) is

$$f_p(R) = R - 2\Lambda . \quad (23)$$

Therefore, we only focus on the homogenous solutions of Eq.(22) hereafter since  $f(R) = f_p(R) - f_h(R)$ . The homogenous part of Eq.(22) is a standard hypergeometric equation

which may have at most 24 solutions in the complex plane around three different singular points ( $R = \infty, 3\Lambda, 4\Lambda$ ) [15]. However, if we focus on real solutions, Eq.(22) may have at most 32 solutions on the real axes around four different singular points  $R = -\infty, 3\Lambda, 4\Lambda, +\infty$ . We will extensively discuss all of these solutions in the following.

The solution around  $+\infty$  reads,

$$f_h(R) = \varpi_1 \left( \frac{\Lambda}{R-4\Lambda} \right)^{p_+-1} {}_2F_1 \left[ q_+, p_+ - 1; r_+; -\frac{\Lambda}{R-4\Lambda} \right] + \varpi_2 \left( \frac{\Lambda}{R-4\Lambda} \right)^{p_- -1} {}_2F_1 \left[ q_-, p_- - 1; r_-; -\frac{\Lambda}{R-4\Lambda} \right] , \quad (24)$$

The above expression is just Eq.(20) which is the physical solution of Eq.(22). We can see clearly that when  $R \rightarrow +\infty$ ,  $f_h(R)$  is well-defined on the real axes and actually  $f_h(R)$  is a real function for the whole range of  $R > 4\Lambda$  as discussed previously.  $R = 4\Lambda$  is a finite point because  $\lim_{R \rightarrow 4\Lambda} f_h(R)$  is a finite value. However, the derivatives  $\frac{d}{dR} f_h(R)$  of all the terms in the above expression are divergent at  $R = 4\Lambda$ .

The solution around  $-\infty$  reads

$$f_h(R) = \varpi_1 \left( \frac{-\Lambda}{R-3\Lambda} \right)^{p_+-1} {}_2F_1 \left[ p_+ - 1, r_+ - q_+; r_+; \frac{\Lambda}{R-3\Lambda} \right] + \varpi_2 \left( \frac{-\Lambda}{R-3\Lambda} \right)^{p_- -1} {}_2F_1 \left[ p_- - 1, r_- - q_-; r_-; \frac{\Lambda}{R-3\Lambda} \right] \quad (25)$$

This expression was obtained in [9]. This solution is well-defined when  $R \rightarrow -\infty$ . It is a real function when  $R < 3\Lambda$ . However, it becomes complex when  $R > 3\Lambda$ .  $R = 3\Lambda$  is a finite point.  $f_h(R)$  and its derivative  $\frac{d}{dR} f_h(R)$  are well defined at  $R = 3\Lambda$ . Clearly Eq.(25) is not a physical solution, it can not satisfy Eq.(6).

The solution around  $3\Lambda$  reads,

$$f_h(R) = \varpi_{12} {}_2F_1 \left[ \alpha_+, \alpha_-; -\frac{1}{2}; \frac{R}{\Lambda} - 3 \right] + \varpi_2 \left( \frac{R}{\Lambda} - 3 \right)^{3/2} {}_2F_1 \left[ \beta_+, \beta_-; \frac{5}{2}; \frac{R}{\Lambda} - 3 \right] , \quad (26)$$

where  $\alpha_{\pm} = (-7 \pm \sqrt{73})/12$  and  $\beta_{\pm} = (11 \pm \sqrt{73})/12$ . The above solution was derived in [10].  $f_h(R)$  and its derivative  $\frac{d}{dR} f_h(R)$  are well-defined on the finite singular point  $R = 3\Lambda$ . When  $R > 4\Lambda$  or  $R < 3\Lambda$ , the second term of Eq.(26) becomes complex. However, in contrast to what was claimed in [10], when  $R > 4\Lambda$  the homogenous part of Eq.(22) do have the real analytical solution, namely Eq.(24).



The solution around  $4\Lambda$  reads,

$$f_h(R) = \varpi_{12} {}_2F_1 \left[ p_+ - 1, p_- - 1; \frac{1}{3}; -\left(\frac{R}{\Lambda} - 4\right) \right] + \varpi_2 \left(\frac{R}{\Lambda} - 4\right)^{2/3} {}_2F_1 \left[ q_+, q_-; \frac{5}{3}; -\left(\frac{R}{\Lambda} - 4\right) \right] \quad (27)$$

Clearly the above solution is well-defined on the finite singular point  $R = 4\Lambda$  and the solution is valid for  $R \geq 4\Lambda$ . It is a solution in the physical range and satisfies Eq.(6). Eq.(27) is equivalent to Eq.(24) since they are defined in the same range. Eq.(27) is simply a new linear combination of the solutions in Eq.(24). However, Eq.(27) has different behaviors at singular points if compared with Eq.(24) owing to the different linear combination of hypergeometric functions. For instance, the derivatives  $\frac{d}{dR}f_h(R)$  of all the terms in Eq.(24) are divergent at  $R = 4\Lambda$  while in Eq.(27), the derivative of the first term  ${}_2F_1 \left[ p_+ - 1, p_- - 1; \frac{1}{3}; -\left(\frac{R}{\Lambda} - 4\right) \right]$  is well-defined at  $R = 4\Lambda$ . Furthermore, Eq.(27) apparently does not have the “chameleon” property because both terms in Eq.(27) are divergent when  $R$  goes to infinity. However, their linear combination, namely, the term in Eq.(21) is convergent when  $R \rightarrow +\infty$ . Therefore, even for equivalent results, we should carefully choose the proper expressions according to boundary conditions at singular points.

Using the Euler transformation and Pfaff transformation, from Eqs.(24,25,26,27), we can find out all the  $8 \times 4 = 32$  real solutions for Eq.(22). Eqs.(24,25,26,27) complete the different behaviors at different singular points for the solutions of Eq.(22). Although Eqs.(24,25,26,27) are substantially different on the real axes, when extended to the complex plane, Eqs.(24,25,26,27) are equivalent to each other because they can be related by connection formulas. However, for different expressions, they have different behaviors at the singular points. It should be very careful to choose the proper expressions according to different boundary conditions. The physical solutions for  $f(R)$  models to describe the Universe should be well-defined in the range  $R > 4\Lambda$  and possess the “chameleon” property  $\lim_{R \rightarrow +\infty} f_h(R) = 0$ ,  $\lim_{R \rightarrow +\infty} \frac{d}{dR}f_h(R) = 0$ . The only physical solution is the “chameleon” part of Eq.(24) ( $\varpi_2 = 0$ ), namely Eq.(21). The result obtained from Eq.(22) has bigger range in the solutions than that obtained from Eq.(6). We need very carefully to pick up the physical solution. It is more convenient to start from Eq.(6) to find physical solutions.

Having the well-defined analytical expression for  $f(R)$  model, we will further discuss its viability. In the very early Universe,  $f_1(R) \sim f_2(R) \sim R$ . In order to evade the instabilities

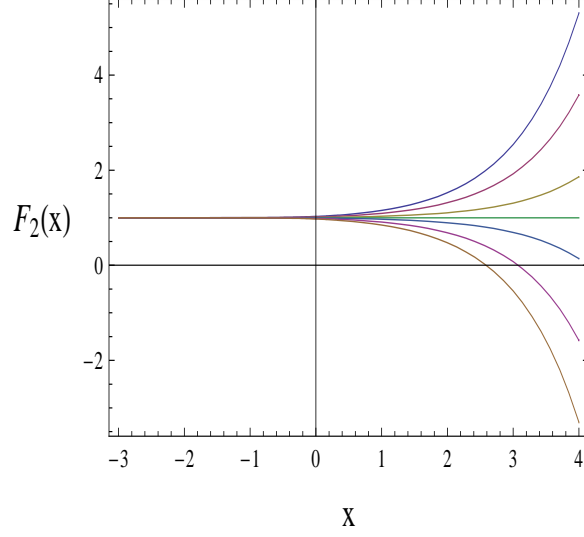


Figure 1: From top  $D_2 = 0.05, 0.03, 0.01, 0, -0.01, -0.03, -0.05$

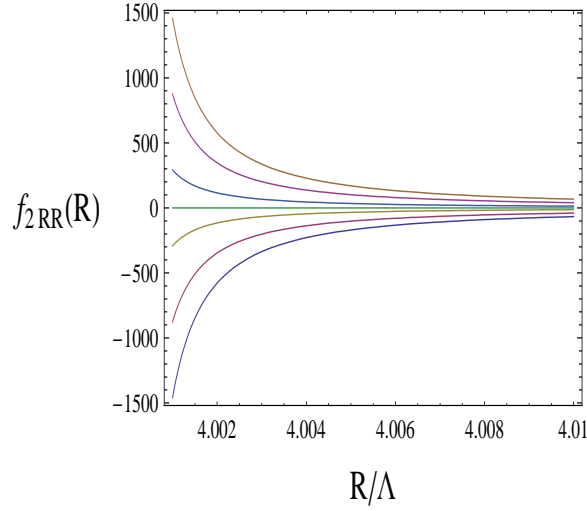


Figure 2: From top  $D_2 = -0.05, -0.03, -0.01, 0, 0.01, 0.03, 0.05$

of  $f(R)$  model, it requires that  $F > 0$  and  $f_{RR} > 0$  [12] [16]. From Fig.(1) and Fig.(2), we can see that when  $D_2 < 0$ , in the past expansion of the Universe  $x < 0$ , we have  $F_2(R) > 0, f_{2RR}(R) > 0$  so that the model  $f_2(R)$  is viable in the past. However, for the future expansion of the Universe, the condition  $D_2 < 0$  can not guarantee  $F_2(x)$  to be always positive. The zero-crossing behavior will lead to singularities in the conformal transformation[17] and the negative  $F$  will lead to the imaginary mass of particles  $\tilde{m} = m/\sqrt{F}$  in the Einstein frame [12] [18]. Furthermore, we can see from Fig.(1) and Fig.(2)

that, when  $x \rightarrow +\infty$  or  $R \rightarrow 4\Lambda$ , the derivative of  $f(R)$ , namely,  $F(R) = \frac{d}{dR}f(R)$  and  $f_{RR}(R) = \frac{d}{dR}F(R)$  are divergent. Although the expression of  $f_2(R)$  has such weakness in the future evolution of the universe,  $f_2(R)$  does not have the future singularity in the Lagrange because  $f_2(R)$  is finite at  $R = 4\Lambda$ . The future point happens at

$$\lim_{x \rightarrow +\infty} R = 4\Lambda \quad , \quad (28)$$

and  $f_2(R)$  is finite at  $R = 4\Lambda$

$$\lim_{R \rightarrow 4\Lambda} f_2(R) = 2\Lambda - \varpi \frac{4(-511 + 79\sqrt{73})\Gamma(2/3)\Gamma(-r_-)}{(-5 + \sqrt{73})(-1 + \sqrt{73})(7 + \sqrt{73})\Gamma(-p_-)\Gamma(q_+)} \approx 2\Lambda - 1.256\varpi \quad , \quad (29)$$

when  $\varpi < 0$ , we can find that  $f_2(4\Lambda) > 2\Lambda$ .

In summary, in this work we have constructed the  $f(R)$  gravity model to mimic the  $\Lambda$ CDM universe expansion in the early and late epochs. We found that there exists a real valued function of Ricci scalar presented in terms of hypergeometric function, which can give rise to the particular cosmological evolution as the  $\Lambda$ CDM model. Although the constructed  $f(R)$  model has weakness in describing the future expansion of the Universe in the Einstein frame, in the Jordan frame it is viable to describe the past evolution of the Universe and it does not have the future singularity in the Lagrange.

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